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A CLASS OF PLASTIC CONSTITUTIVE EQUATIONS WITH VERTEX EFFECT—I. GENERAL THEORY

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Abstract—A class of plastic constitutive equations which shows one-to-one correspondence between plastic strain increment $d\boldsymbol{\epsilon}^p$ and stress increment $d\boldsymbol{\sigma}$ is proposed from the viewpoint of tensor algebra. It is shown that it inevitably represents the so-called vertex-hardening model. Several examples of plastic constitutive equation among this class are formulated. For plasticity, the stress time-like measure $\overline{d\sigma} = [(3/2)\,\mathrm{tr}(\overset{\circ}{d}\mathbf{T}^2)]^{1/2}$ and the strain time-like measure $\overline{d\epsilon} = [(2/3)\,\mathrm{tr}(d\mathbf{e}^p)^2]^{1/2}$ are effectively used to represent loading or straining history, where $\overset{\circ}{d}\mathbf{T}$ is the increment of deviatoric stress in the sense of Jaumann's rate, and $d\mathbf{e}^p$ denotes the plastic deviatoric strain increment. First, material is considered to be initially isotropic but to be anisotropic with deformation. For this case, Wang's representation theorem on isotropic tensor functions is effectively used. However, anisotropy in this case is limited. Therefore the theory is extended to the case where general initial and subsequent anisotropy plays an important role. Then it becomes possible that any anisotropic rule of yielding such as kinematic hardening, kinematic-isotropic hardening and other general anisotropic hardening without vertex formation is combined with vertex hardening. Introducing the natural time-measure dt, the theory is extended to express natural time-dependent inelastic constitutive equations such as creep and/or viscoelasticity. Furthermore, introducing the natural and internal time-measures, which are combinations of dt and $\overline{d\sigma}$, or dt and $\overline{d\epsilon^i}$, the theory is also extended to the case where natural and internal time-dependent inelastic constitutive equations such as viscoplasticity and/or dynamic plasticity are required to formulate. In some cases, temperature dependency is also considered.

1. INTRODUCTION

The plastic constitutive equation is usually understood as a relation between plastic strain increment $d\boldsymbol{\epsilon}^p$ and the corresponding stress $\boldsymbol{\sigma}$ with appropriate history dependence. However, it is also well known that, in plastic instability phenomena, such as buckling under compression, necking under tension or shear band formation under compression or tension, the critical load predicted is very often far larger than the experimental value. This implies that the effect of the stress increment $d\boldsymbol{\sigma}$ cannot be neglected for the case where the strain- (or loading) path is abruptly changed or deviates severely from the proportional loading.

Sewell[1] uses three hyperplanes as a loading surface in stress space to express the effect of $d\boldsymbol{\sigma}$ following Koiter's suggestion of multiyield surface[2]. Itoh[3] proposed a plastic constitutive equation including $d\boldsymbol{\sigma}$ from the polycrystalline point of view. Stören and Rice[4] used the incremental form of J_2-deformation theory as a vertex-hardening model. Christoffersen and Hutchinson[5] proposed a group of plastic constitutive equation with vertex effect considering it as a relation between $d\boldsymbol{\epsilon}^p$ and $d\boldsymbol{\sigma}$ introducing the plastic potential as a function of $d\boldsymbol{\sigma}$. Its simplest form is now called J_2-corner theory and applied to several bifurcation problems.

Here, with the presupposition of one-to-one correspondence between $d\boldsymbol{\epsilon}^p$ and $d\boldsymbol{\sigma}$, we present the general form of plastic constitutive equation with vertex effect satisfying mathematical restriction about tensor functions. It will give a clear means to develop further the more strict vertex-hardening model. The theory is extended to that for the case where initial and subsequent general anisotropy plays an important role. Furthermore, time-dependent inelastic constitutive equations are also discussed and formulated.

More concrete and extensive discussions about the simplest form among the constitutive equations proposed here and its applications to some bifurcation problems will appear in the following works of this series.

2. BASIC EQUATIONS

The general form of the plastic constitutive equation consisting of $d\boldsymbol{\epsilon}^p$, $d\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}^p$ and $\boldsymbol{\sigma}$ is written formally as follows:

$$d\boldsymbol{\epsilon}^p = \mathbf{E}(\mathring{d}\boldsymbol{\sigma}, \boldsymbol{\sigma}, \boldsymbol{\epsilon}^p; \epsilon), \tag{1}$$

or in its inverse form as

$$\mathring{d}\boldsymbol{\sigma} = \mathbf{S}(d\boldsymbol{\epsilon}^p, \boldsymbol{\sigma}, \boldsymbol{\epsilon}^p; \epsilon), \tag{2}$$

where

$\boldsymbol{\sigma}$: Cauchy stress tensor,
$d\boldsymbol{\epsilon}^p$: Euler's plastic strain increment tensor,
$\boldsymbol{\epsilon}^p = \boldsymbol{\epsilon} - \boldsymbol{\epsilon}^e$,
$\boldsymbol{\epsilon}$: total strain tensor,
$\boldsymbol{\epsilon}^e$: elastic strain tensor,
$\epsilon = \int \overline{d\epsilon^p}$,
$\overline{d\epsilon^p} = \sqrt{2/3}[\mathrm{tr}(d\boldsymbol{\epsilon}^{p\prime})^2]^{1/2}$,
$de^{p} = d\boldsymbol{\epsilon}^p - (1/3)(\mathrm{tr}\, d\boldsymbol{\epsilon}^p)\mathbf{I}$,
$\mathbf{I}$: unity tensor of second rank,
tr: "trace" symbol,
$\mathring{d}\boldsymbol{\sigma} = d\boldsymbol{\sigma} - d\boldsymbol{\omega}\boldsymbol{\sigma} + \boldsymbol{\sigma}\, d\boldsymbol{\omega}$ = Jaumann stress increment tensor,
$d\boldsymbol{\omega}$: incremental rigid-body rotation tensor.

Here we concern ourselves with the restrictive form of constitutive equation as a relation between $d\boldsymbol{\epsilon}^p$ and $d\boldsymbol{\sigma}$ which can be reduced from the following two expressions by eliminating $\boldsymbol{\epsilon}^p$ from them:

$$\begin{aligned} d\boldsymbol{\epsilon}^p &= \mathbf{E}^*(\boldsymbol{\sigma}, \boldsymbol{\epsilon}^p; \epsilon), \\ \mathring{d}\boldsymbol{\sigma} &= \mathbf{S}^*(\boldsymbol{\sigma}, \boldsymbol{\epsilon}^p; \epsilon). \end{aligned} \tag{3}$$

For convenience, we introduce the time-like measure $\overline{d\sigma}$ (>0) for description of stress-history by the definition

$$\overline{d\sigma} = \sqrt{3/2}[\mathrm{tr}(\mathring{d}\mathbf{T})^2]^{1/2}, \tag{4}$$

where $\mathbf{T}$ denotes the deviatoric stress tensor. This definition is analogical to that of Mises' equivalent stress $\bar{\sigma}$, i.e.

$$\bar{\sigma} = \sqrt{3/2}(\mathrm{tr}\, \mathbf{T}^2)^{1/2}. \tag{5}$$

We should note that the derivative of $\bar{\sigma}$, $d\bar{\sigma}$, is expressed as

$$d\bar{\sigma} = (3/2\bar{\sigma})\, \mathrm{tr}(\mathbf{T}\, \mathring{d}\mathbf{T}) \qquad (\gtreqless 0) \tag{6}$$

which is not generally equal to the measure $\overline{d\sigma}$ except for the case where $\mathring{d}\mathbf{T}$ is always parallel to $\mathbf{T}$ throughout loading history.

Now we define the nondimensional stress-rate or deviatoric stress-rate as

$$\mathring{\boldsymbol{\sigma}} = \mathring{d}\boldsymbol{\sigma}/\overline{d\sigma} \qquad \text{or} \qquad \mathring{\mathbf{T}} = \mathring{d}\mathbf{T}/\overline{d\sigma}, \tag{7}$$

from which we find easily the equality

$$\mathrm{tr}\, \mathring{\mathbf{T}}^2 = 2/3. \tag{8}$$

On the other hand, we introduce another time-like measure $\overline{d\epsilon^p}$ (>0) for the description of strain-history by the definition

$$\overline{d\epsilon^p} = \sqrt{2/3}[\mathrm{tr}(d\boldsymbol{\epsilon}^p)^2]^{1/2}. \tag{9}$$

Using this measure, the plastic strain-rate and its deviator are defined as

$$\dot{\boldsymbol{\epsilon}}^p = d\boldsymbol{\epsilon}^p/\overline{d\epsilon^p}, \qquad \dot{\mathbf{e}}^p = d\mathbf{e}^p/\overline{d\epsilon^p}, \tag{10}$$

from which we get

$$\mathrm{tr}(\dot{\mathbf{e}}^p)^2 = 3/2. \tag{11}$$

Now the equation in eqn (3) are rewritten as follows:

$$\dot{\boldsymbol{\epsilon}}^p = \mathbf{E}^{**}(\boldsymbol{\sigma}, \boldsymbol{\epsilon}^p; \epsilon); \qquad \mathring{\boldsymbol{\sigma}} = \mathbf{s}^{**}(\boldsymbol{\sigma}, \boldsymbol{\epsilon}^p; \epsilon). \tag{12}$$

If the material can be assumed to be initially isotropic, then, with the aid of Wang's theorem[6] on isotropic tensor functions, we can represent exactly the two equations in eqn (12) in terms of polynomials. Namely,

$$\mathrm{tr}\, \mathring{\boldsymbol{\sigma}} = \eta_0, \tag{13}$$

$$\begin{aligned}\mathring{\mathbf{T}} = {} & \eta_1 \mathbf{T} + \eta_2[\mathbf{T}^2 - (1/3)(\mathrm{tr}\, \mathbf{T}^2)\mathbf{I}] + \eta_3 \mathbf{e}^p + \eta_4[\mathbf{e}^{p2} - (1/3)(\mathrm{tr}\, \mathbf{e}^{p2})\mathbf{I}] \\ & + \eta_5[\mathbf{T}\mathbf{e}^p + \mathbf{e}^p\mathbf{T} - (2/3)(\mathrm{tr}\, \mathbf{T}\mathbf{e}^p)\mathbf{I}] + \eta_6[\mathbf{T}^2\mathbf{e}^p + \mathbf{e}^p\mathbf{T}^2 \\ & - (2/3)(\mathrm{tr}\, \mathbf{T}^2\mathbf{e}^p)\mathbf{I}] + \eta_7[\mathbf{T}\mathbf{e}^{p2} + \mathbf{e}^{p2}\mathbf{T} - (2/3)(\mathrm{tr}\, \mathbf{T}\mathbf{e}^{p2})\mathbf{I}],\end{aligned} \tag{14}$$

$$\begin{aligned}\eta_i = {} & \eta_i(\mathrm{tr}\, \boldsymbol{\sigma}, \mathrm{tr}\, \mathbf{T}^2, \mathrm{tr}\, \mathbf{T}^3, \mathrm{tr}\, \boldsymbol{\epsilon}^p, \mathrm{tr}\, \mathbf{e}^{p2}, \mathrm{tr}\, \mathbf{e}^{p3}, \mathrm{tr}\, \mathbf{T}\mathbf{e}^p, \\ & \mathrm{tr}\, \mathbf{T}^2\mathbf{e}^p, \mathrm{tr}\, \mathbf{T}\mathbf{e}^{p2}, \mathrm{tr}\, \mathbf{T}^2\mathbf{e}^{p2}; \epsilon),\end{aligned} \tag{15}$$

where $i = 0, 1, \ldots, 7$. And

$$\mathrm{tr}\, \dot{\boldsymbol{\epsilon}}^p = \zeta_0, \tag{16}$$

$$\begin{aligned}\dot{\mathbf{e}}^p = {} & \zeta_1 \mathbf{T} + \zeta_2[\mathbf{T}^2 - (1/3)(\mathrm{tr}\, \mathbf{T}^2)\mathbf{I}] + \zeta_3 \mathbf{e}^p + \cdots + \zeta_7[\mathbf{T}\mathbf{e}^{p2} \\ & + \mathbf{e}^{p2}\mathbf{T} - (2/3)(\mathrm{tr}\, \mathbf{T}\mathbf{e}^{p2})\mathbf{I}],\end{aligned} \tag{17}$$

$$\zeta_i = \zeta_i(\mathrm{tr}\, \boldsymbol{\sigma}, \mathrm{tr}\, \mathbf{T}^2, \ldots, \mathrm{tr}\, \mathbf{T}^2\mathbf{e}^{p2}; \epsilon), \qquad i = 0, 1, \ldots, 7. \tag{18}$$

η_i and ζ_i could involve $\mathring{\mathbf{T}}$ and $\dot{\mathbf{e}}^p$ as 0-th order homogeneous terms.

From eqns (14) and (8), and from eqns (17) and (11), we obtain the following two scalar relations between $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}^p$, which can be thought as the so-called yield conditions:

$$\begin{aligned} 1 &= \phi_1(\mathrm{tr}\, \boldsymbol{\sigma}, \mathrm{tr}\, \mathbf{T}^2, \ldots, \mathrm{tr}\, \mathbf{T}^2\mathbf{e}^{p2}; \epsilon), \\ 1 &= \phi_2(\mathrm{tr}\, \boldsymbol{\sigma}, \mathrm{tr}\, \mathbf{T}^2, \ldots, \mathrm{tr}\, \mathbf{T}^2\mathbf{e}^{p2}; \epsilon), \end{aligned} \tag{19}$$

both of which should be identical to each other. Equation (19) proves the existence of the yield condition for a plastic material which is initially isotropic.

The expressions (13)–(18) are exact in a mathematical sense. Additionally, they must obey the physical condition—i.e. the second law of thermodynamics, which is expressed as follows:

$$\mathrm{tr}(\mathbf{T}\dot{\mathbf{e}}^p) + \boldsymbol{\mu}\cdot\dot{\boldsymbol{\alpha}} \geqq 0, \tag{20}$$

$$(\mathrm{tr}\, \boldsymbol{\sigma})(\mathrm{tr}\, \dot{\mathbf{e}}^p) + (\mathrm{tr}\, \boldsymbol{\mu})(\mathrm{tr}\, \dot{\boldsymbol{\alpha}}) \geqq 0, \tag{21}$$

where $\boldsymbol{\mu}$ and $\dot{\boldsymbol{\alpha}}$ are the generalized force and the generalized flux associated with the internal state which are neglected here.

By eliminating $\mathbf{e}^p$ from eqns (14) and (17), we can obtain a class of plastic constitutive equations in terms of $\dot{\mathbf{e}}^p$, $\mathring{\boldsymbol{\sigma}}$, and $\boldsymbol{\sigma}$ with the effect of $\boldsymbol{\epsilon}^p$ and $\boldsymbol{\epsilon}$ (or strain history) through the fundamental invariants such as tr $\boldsymbol{\epsilon}^p$, tr $\mathbf{e}^{p2}$, tr $\mathbf{e}^{p3}$, tr $\mathbf{e}^p\mathbf{T}$, tr $\mathbf{e}^{p2}\mathbf{T}$, tr $\mathbf{e}^p\mathbf{T}^2$ and tr $\mathbf{e}^{p2}\mathbf{T}^2$.

3. REDUCED CONSTITUTIVE EQUATIONS

First, for dilatational part of the constitutive equation, we can easily obtain the following relation from eqns (13) and (16):

$$\begin{aligned} \operatorname{tr}(d\boldsymbol{\epsilon}^p) &= \zeta^* \operatorname{tr}(d\boldsymbol{\sigma}), \\ \zeta^* &= \zeta^*(\operatorname{tr}\boldsymbol{\sigma}, \operatorname{tr}\mathbf{T}^2, \ldots; \boldsymbol{\epsilon}). \end{aligned} \tag{22}$$

If the incompressible plasticity is considered, ζ^* is identically equal to 0.

For deviatoric part of the constitutive equation, we can make various reduced forms from eqns (14) and (17).

3.1 *First-rank approximation—I*

Adopting the first two terms of the right-hand sides in eqns (14) and (17), i.e.

$$\begin{aligned} \mathring{\mathbf{T}} &= \eta_1\mathbf{T} + \eta_3\mathbf{e}^p, \\ \dot{\mathbf{e}}^p &= \zeta_1\mathbf{T} + \zeta_3\mathbf{e}^p, \end{aligned} \tag{23}$$

and eliminating $\mathbf{e}^p$ from them, we have the simplest form of the plastic constitutive equation within the framework of our idea as follows:

$$\dot{\mathbf{e}}^p = \alpha_1\mathbf{T} + \alpha_2\mathring{\mathbf{T}}, \tag{24}$$

$$\alpha_1 = \zeta_1 - \eta_1\alpha_2, \qquad \alpha_2 = \zeta_3/\eta_3, \tag{25}$$

$$\alpha_i = \alpha_i(\operatorname{tr}\boldsymbol{\sigma}, \operatorname{tr}\mathbf{T}^2, \ldots, \operatorname{tr}\mathbf{T}^2\mathbf{e}^{p2}; \boldsymbol{\epsilon}), \qquad i = 1, 2. \tag{26}$$

Let us concern ourselves with only strain-hardening materials. Then the following equi-inequalities should hold:

$$\operatorname{tr}(\dot{\mathbf{e}}^p\mathring{\mathbf{T}}) \geqq 0 \quad \text{and} \quad \operatorname{tr}(\dot{\mathbf{e}}^p\mathbf{T}) \geqq 0. \tag{27}$$

Making scalar products of eqn (24) with $\mathbf{T}$ and $\mathring{\mathbf{T}}$, and using eqn (8), eqn (27) yields

$$\begin{aligned} \alpha_1 \operatorname{tr}(\mathbf{T}\mathring{\mathbf{T}}) + (2/3)\alpha_2 &\geqq 0, \\ \alpha_1 \operatorname{tr}\mathbf{T}^2 + \alpha_2 \operatorname{tr}(\mathbf{T}\mathring{\mathbf{T}}) &\geqq 0. \end{aligned} \tag{28}$$

These would hold even for the case where $\mathbf{T}$ is perpendicular to $\mathring{\mathbf{T}}$, or in other words, $\operatorname{tr}(\mathbf{T}\mathring{\mathbf{T}}) = 0$, then $\alpha_1 \geqq 0$ and $\alpha_2 \geqq 0$.

Now let us introduce the angle Θ defined by the following relation:

$$\begin{aligned} \cos\Theta &= \operatorname{tr}(\mathbf{T}\mathring{\mathbf{T}})/[(\operatorname{tr}\mathbf{T}^2)(\operatorname{tr}\mathring{\mathbf{T}}^2)]^{1/2} \\ &= (3/2)\operatorname{tr}(\mathbf{T}\, d\mathring{\mathbf{T}})/(\bar{\sigma}\, \overline{d\sigma}). \end{aligned} \tag{29}$$

It is easily confirmed that Θ so defined is the angle between $\mathbf{T}$ and $\mathring{\mathbf{T}}$ in the so-called

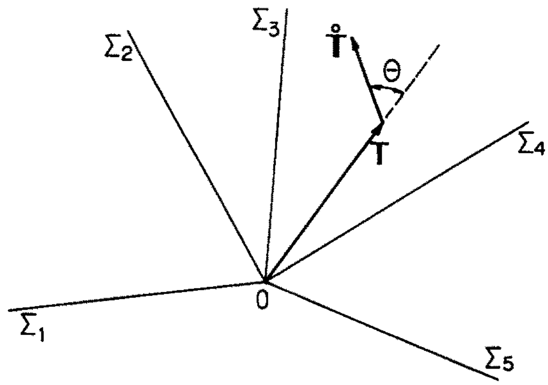

Fig. 1. **T** and $\mathring{\mathbf{T}}$ in the five-dimensional deviatoric stress space.

Illushin's five-dimensional deviatoric stress space **Σ**[7], where

$$\begin{aligned} \boldsymbol{\Sigma} &= (\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5), \\ \Sigma_1 &= (3/2)T_{11}, \quad \Sigma_2 = \sqrt{3}[(T_{11}/2) + T_{22}], \\ \Sigma_3 &= \sqrt{3}\,T_{12}, \quad \Sigma_4 = \sqrt{3}\,T_{23}, \quad \Sigma_5 = \sqrt{3}\,T_{31}, \end{aligned} \tag{30}$$

(see Fig. 1).

From eqns (28) and (29), we find

$$\cos\Theta \geqq \mathrm{Max}[-(\alpha_2/\alpha_1\bar{\sigma}),\ -(\alpha_1\bar{\sigma}/\alpha_2)]. \tag{31}$$

The second term of this right-hand side corresponds to the case where $\dot{\mathbf{e}}^p$ is set **0** in eqn (24) and thus $\mathring{\mathbf{T}} = -(\alpha_2/\alpha_1)\mathbf{T}$, which means pure unloading because $\mathring{\mathbf{T}}$ points to the inverse direction to **T**. Therefore, we understand that the first term in the right-hand side of eqn (31) makes sense, which is associated with the condition $\mathrm{tr}(\dot{\mathbf{e}}^p\mathring{\mathbf{T}}) \geqq 0$. Consequently,

$$\cos\Theta \geqq -(\alpha_2/\alpha_1\bar{\sigma}). \tag{32}$$

The equality in this relation holds for the case $\mathrm{tr}(\dot{\mathbf{e}}^p\mathring{\mathbf{T}}) = 0$, i.e. $\mathring{\mathbf{T}}$ makes a right angle to $\dot{\mathbf{e}}^p$ in **Σ**-space.

Equation (32) implies that the loading surface possesses a vertex at the loading point if α_2 is not equal to 0. That is, the loading surface generally forms a circular cone with **T** as its axis, and the half angle of the cone Θ_0 is given by the equation

$$\Theta_0 = \pi - \Theta_{max}, \tag{33}$$

$$\Theta_{max} = \cos^{-1}[-(\alpha_2/\alpha_1\bar{\sigma})] \geqq \pi/2. \tag{34}$$

Plastic loading continues for the subsequent loading increment $\mathring{\mathrm{d}}\mathbf{T}$ for Θ within the range

$$0 \leqq \Theta \leqq \Theta_{max}. \tag{35}$$

The direction of $\dot{\mathbf{e}}^p$ lies within the cone which makes a right angle to the loading cone (see Fig. 2).

The constitutive equation (24) is rewritten in an incremental form as follows:

$$\mathrm{d}\mathbf{e}^p = \alpha_1\mathbf{T}\,\overline{\mathrm{d}\epsilon^p} + (\alpha_2/3h^*)\,\mathring{\mathrm{d}}\mathbf{T}, \qquad 3h^* = \overline{\mathrm{d}\sigma}/\overline{\mathrm{d}\epsilon^p}. \tag{36}$$

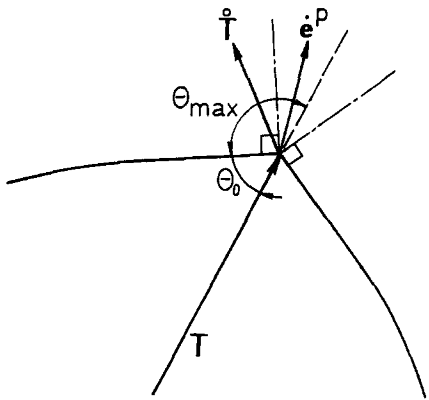

Fig. 2. Θ_{max} and Θ_0, and range of the direction of $\dot{e}^p$ at the vertexed loading point.

Taking eqn (25) into consideration, and setting as

$$\eta_1 = 1/\bar{\sigma}, \qquad \zeta_1 = (3/2\bar{\sigma})(h^*/h), \qquad \eta_3 = 1/3h^*, \qquad \zeta_3 = 1/2H, \tag{37}$$

we can rewrite eqn (36) as follows:

$$d\mathbf{e}^p = \frac{1}{2\bar{\sigma}}\left(\frac{1}{h} - \frac{1}{H}\right)\mathbf{T}\,\overline{d\sigma} + \frac{1}{2H}\,\mathring{d}\mathbf{T}, \tag{38}$$

which is similar to the version of extended hypoelasticity of the incremental form of J_2-deformation theory[4], i.e.

$$d\mathbf{e}^p = \frac{1}{2\bar{\sigma}}\left(\frac{1}{h'} - \frac{1}{H'}\right)\mathbf{T}\left(\frac{3}{2\bar{\sigma}}\,\mathrm{tr}(\mathbf{T}\,\mathring{d}\mathbf{T})\right) + \frac{1}{2H'}\,\mathring{d}\mathbf{T}, \tag{39}$$

where $3h' = d\bar{\sigma}/d\bar{\epsilon}^p$. h' and H' are called the instantaneous hardening rate and the instantaneous vertex-hardening rate, respectively. Hereafter, we also call h and H in eqn (38) in the same way.

Now we should note that h and H in eqn (38) are dependent on Θ. Therefore we set as follows:

$$h = h_0/P(\Theta), \qquad H = H_0/P(\Theta), \tag{40}$$

where $P(0) = 1$, and thus h_0 and H_0 are h and H for the case of $\Theta = 0$, respectively. Equation (38) becomes

$$d\mathbf{e}^p = \langle P(\Theta)\rangle\{(1/2\bar{\sigma})[(1/h_0) - (1/H_0)]\mathbf{T}\,\overline{d\sigma} + (1/2H_0)\,\mathring{d}\mathbf{T}\}, \tag{41}$$

where $\langle P\rangle = P$ for $P > 0$, and $\langle P\rangle = 0$ for $P \leqq 0$.

In order to determine the function $P(\Theta)$, we make the scalar product $\mathrm{tr}(d\mathbf{e}^p\,\mathring{d}\mathbf{T})$ for eqn (41). Then we get

$$\mathrm{tr}(d\mathbf{e}^p\,\mathring{d}\mathbf{T}) = (\overline{d\sigma}^2/3h_0)P(\Theta)(a + b\cos\Theta), \tag{42}$$

$$a = h_0/H_0, \qquad b = 1 - a. \tag{43}$$

Equation (42) should be positive or equal to 0 because of eqn (27). Noting $h_0 > 0$, $P(\Theta)$ should have the form

$$P(\Theta) = c(a + b\cos\Theta)^m, \tag{44}$$

where $c > 0$ and m = an odd integer. Moreover, it is natural to think that eqn (41) reduces to J_2-flow theory when H_0 tends to infinity ($H_0 \to \infty$). If $H_0 \to \infty$, eqn (41) reduces to

$$d\mathbf{e}^p = P(\Theta)(1/2\bar{\sigma}h_0)\mathbf{T}\,\overline{d\sigma}, \tag{45}$$

where $\langle\ \rangle$ is omitted. On the other hand, J_2-flow theory is written as $d\mathbf{e}^p = (3/4\bar{\sigma}^2 h_0)\mathbf{T}$ tr($\mathbf{T}\,\mathring{d}\mathbf{T}$). Therefore, taking eqn (29) into consideration, eqn (45) coincides with J_2-flow theory if $P(\Theta) = \cos\Theta$ in eqn (45). For that, $c = 1$ and $m = 1$ in eqn (44). Eventually, the most reasonable form of the function $P(\Theta)$ is given as follows:

$$P(\Theta) = a + b\cos\Theta. \tag{46}$$

The constitutive equation (41) together with eqn (46) satisfies the following conditions:

(1) For proportional loading, it reduces to the incremental form of J_2-deformation theory expressed by eqn (39), which is now understood to be a kind of extended hypoelastic constitutive equations[8].

(2) $\dot{\mathbf{e}}^p = \mathbf{0}$ for $\Theta = \Theta_{max}$.

(3) As described above, it reduces to J_2-flow theory when $H_0 \to \infty$ and thus $\Theta_0 = \Theta_{max} = \pi/2$ (no vertex).

We can easily confirm that the constitutive equation (41) together with eqn (46) yields a unique inverse relation.

More detailed discussions and applications of eqn (41) with eqn (46) will appear in the following papers of the series of this work.

Finally, if we consider the case where tr($\dot{\mathbf{e}}^p\mathring{\mathbf{T}}$) < 0 and tr($\dot{\mathbf{e}}^p\mathbf{T}$) ≧ 0, eqn (24) can express the constitutive equation with work-softening. In this case, we have a concave vertex on the loading surface at the loading point.

3.2 *First-rank approximation—II*

Let us adopt the following expressions in eqns (14) and (17):

$$\begin{aligned}\mathring{\mathbf{T}} &= \eta_1\mathbf{T} + \eta_3\mathbf{e}^p + \eta_5[\mathbf{T}\mathbf{e}^p + \mathbf{e}^p\mathbf{T} - (2/3)(\mathrm{tr}\,\mathbf{T}\mathbf{e}^p)\mathbf{I}],\\ \dot{\mathbf{e}}^p &= \zeta_1\mathbf{T} + \zeta_3\mathbf{e}^p + \zeta_5[\mathbf{T}\mathbf{e}^p + \mathbf{e}^p\mathbf{T} - (2/3)(\mathrm{tr}\,\mathbf{T}\mathbf{e}^p)\mathbf{I}].\end{aligned} \tag{47}$$

Eliminating $\dot{\mathbf{e}}^p$ from these equations, we obtain

$$\dot{\mathbf{e}}^p = \alpha_2\mathbf{K}:\mathring{\mathbf{T}} + \alpha_1\mathbf{M}:\mathbf{T} + \alpha_3\mathbf{M}:[\mathbf{T}^2 - (1/3)(\mathrm{tr}\,\mathbf{T}^2)\mathbf{I}], \tag{48}$$

where

$$\alpha_1,\alpha_2 = [\text{see eqn (25)}], \qquad \alpha_3 = 2(\eta_5\zeta_1 - \eta_1\zeta_5)/\eta_3,$$
$$\mathbf{K} = \mathbf{M}:[\mathbf{g} + 2\xi_2(\mathbf{L}'/\bar{\sigma})], \qquad \mathbf{M} = [\mathbf{g} + 2\xi_1(\mathbf{L}'/\bar{\sigma})]^{-1},$$
$$(\mathbf{L}')_{ijkl} = (1/2)(\delta_{ij}T_{kl} + \delta_{jl}T_{ik}) - (1/3)\delta_{ij}T_{kl},$$
$$\xi_1 = 2\eta_5^*/\eta_3, \qquad \xi_2 = 2\zeta_5^*/\zeta_3, \qquad \eta_5^* = \bar{\sigma}\eta_5, \qquad \zeta_5^* = \bar{\sigma}\zeta_5,$$

and $\mathbf{g}$ = unity tensor of fourth rank, δ_{ij} = Kronecker's delta, and : denotes double products, i.e. $(\mathbf{M}:\mathbf{L}')_{ijkl} = M_{ijmn}L'_{nmkl}$ where the summation convention is adopted. By the similar substitution as in eqn (37), we get

$$\begin{aligned}\alpha_3 &= P\beta\eta_1(1/\bar{\sigma}^2)[(2\zeta_1/\gamma\eta_1)(1/h_0) - (1/H_0)],\\ \beta &= (3/4)\xi_2, \qquad \gamma = 3\xi_2/\xi_1.\end{aligned} \tag{49}$$

Setting $\eta_1 = \eta/\bar{\sigma}$, $\eta \simeq 1$, we get

$$\alpha_1 = P(3\eta/4\bar{\sigma}^2)[\mu_1(1/h_0) - (1/H_0)], \qquad \mu_1 = 2\zeta_1/3\eta_1 \simeq 1,$$
$$\alpha_3 = P(3\eta\xi_2/4\bar{\sigma}^3)[\mu_2(1/h_0) - (1/H_0)], \qquad \mu_2 = \mu_1(\xi_1/\xi_2).$$

In a special case where $\xi_1 = \xi_2$, then $\mu_1 = \mu_2 \simeq 1$, $\alpha_3 = \xi_1\alpha_1/\bar{\sigma}$, and

$$\alpha_1 \simeq P(3\eta/4\bar{\sigma}^2)[(1/h_0) - (1/H_0)]; \qquad \mathbf{K} = \mathbf{g}. \tag{50}$$

Equation (48) reduces to

$$\dot{\mathbf{e}}^p = \alpha_2\mathring{\mathbf{T}} + \alpha_1\mathbf{M}:\mathbf{T}^*, \tag{51}$$
$$\mathbf{T}^* = \mathbf{T} + (\alpha_3/\alpha_1)[\mathbf{T}^2 - (1/3)(\mathrm{tr}\,\mathbf{T}^2)\mathbf{I}], \tag{52}$$

where $\mathbf{T}^*$ is parallel to $\mathbf{T}$.

If the material is hardening, $\mathrm{tr}(\dot{\mathbf{e}}^p\mathring{\mathbf{T}}) \geqq 0$, then

$$\begin{aligned}\mathrm{tr}(\mathbf{T}^{**}\mathring{\mathbf{T}}) &\geqq -(\alpha_2/\alpha_1)\,\mathrm{tr}\,\mathring{\mathbf{T}}^2 = -(2/3)\alpha_2/\alpha_1,\\ \mathbf{T}^{**} &= \mathbf{M}:\mathbf{T}^*.\end{aligned} \tag{53}$$

Introducing the angle Θ^* between $\mathbf{T}^{**}$ and $\mathring{\mathbf{T}}$, we see

$$\cos\Theta^* = \mathrm{tr}(\mathbf{T}^{**}\mathring{\mathbf{T}})/[\sqrt{2/3}(\mathrm{tr}\,\mathbf{T}^{**2})^{1/2}] \geqq -(\mathrm{tr}\,\mathbf{T}^2/\mathrm{tr}\,\mathbf{T}^{**2})^{1/2}/[\eta(\mu_1H_0/h_0 - 1)], \tag{54}$$

which implies that the loading surface forms a convex cone with its vertex at the loading point and with $\mathbf{T}^{**}$ as its axis. The smaller H_0 is, the more acute the cone is.

3.3 *First-rank approximation—III*

Let us adopt the following expressions in eqns (14) and (17):

$$\begin{aligned}\mathring{\mathbf{T}} &= \eta_1\mathbf{T} + \eta_2[\mathbf{T}^2 - (1/3)(\mathrm{tr}\,\mathbf{T}^2)\mathbf{I}] + \eta_3\mathbf{e}^p + \eta_5\mathbf{P},\\ \dot{\mathbf{e}}^p &= \zeta_1\mathbf{T} + \zeta_2[\mathbf{T}^2 - (1/3)(\mathrm{tr}\,\mathbf{T}^2)\mathbf{I}] + \zeta_3\mathbf{e}^p + \zeta_5\mathbf{P},\\ \mathbf{P} &= \mathbf{T}\mathbf{e}^p + \mathbf{e}^p\mathbf{T} - (2/3)(\mathrm{tr}\,\mathbf{T}\mathbf{e}^p)\mathbf{I}.\end{aligned} \tag{55}$$

Eliminating $\mathbf{e}^p$ and neglecting the term $\mathbf{T}^3$, we obtain the same equation as in the Section 3.2 with replacement of $2(\eta_1\zeta_5 - \eta_5\zeta_1)$ by $2(\eta_1\zeta_5 - \eta_5\zeta_1) + (\eta_2\zeta_3 - \eta_3\zeta_2)$. Therefore α_3 is replaced by

$$\alpha_3 = P\{(3\eta\xi_2/4\bar{\sigma}^3)[\mu_2(1/h_0) - (1/H_0)] + (3\eta_2/4\bar{\sigma})[(2\zeta_2/3\eta_2)(1/h_0) - (1/H_0)]\}. \tag{56}$$

3.4 *First-rank approximation—IV*

Let us consider the case where $\eta_4 = \eta_7 = 0$ and $\zeta_4 = \zeta_7 = 0$ in eqns (14) and (17). Then eliminating $\mathbf{e}^p$ from them, we obtain the following constitutive equation:

$$\dot{\mathbf{e}}^p = \alpha_2\mathbf{K}_1:\mathring{\mathbf{T}} + (\zeta_1\mathbf{g} - \eta_1\alpha_2\mathbf{K}_1):\mathbf{T} + (\zeta_2\mathbf{g} - \eta_2\alpha_2\mathbf{K}_1):[\mathbf{T}^2 - (1/3)(\mathrm{tr}\,\mathbf{T}^2)\mathbf{I}], \tag{57}$$

$$\begin{aligned}\mathbf{K}_1 &= \mathbf{B}:\mathbf{A}^{-1}, \qquad \mathbf{A} = \mathbf{g} + (2\eta_5/\eta_3)\mathbf{L}' + (2\eta_6/\eta_3)\mathbf{L}_1,\\ \mathbf{B} &= \mathbf{g} + (2\zeta_5/\zeta_3)\mathbf{L}' + (2\zeta_6/\zeta_3)\mathbf{L}_1,\\ (\mathbf{L}_1)_{ijkl} &= (1/2)(T_{im}T_{ml}\delta_{kj} + T_{km}T_{mj}\delta_{li}) - (1/3)T_{km}T_{ml}\delta_{ij}.\end{aligned} \tag{58}$$

Again a convex cone can be confirmed to form at the loading point on the loading surface, though further discussion is omitted here.

3.5 *Quadratic constitutive equation*

No approximation is made here. Equations (14) and (17) are rewritten as follows:

$$\mathring{\mathbf{T}} = \eta_1\mathbf{T} + \eta_2\mathbf{T}^* + \eta_3\mathbf{A}:\mathbf{e}^p + \eta_4\mathbf{C}_1:\mathbf{e}^{p2}, \tag{59}$$

$$\dot{\mathbf{e}}^p = \zeta_1\mathbf{T} + \zeta_2\mathbf{T}^* + \zeta_3\mathbf{B}:\mathbf{e}^p + \zeta_4\mathbf{C}_2:\mathbf{e}^{p2}, \tag{60}$$

where

$$\mathbf{T}^* = \mathbf{T}^2 - (1/3)(\mathrm{tr}\,\mathbf{T}^2)\mathbf{I}, \tag{61}$$

$$\mathbf{C}_1 = \mathbf{g}' + (2\eta_7/\eta_4)\mathbf{L}', \qquad \mathbf{C}_2 = \mathbf{g}' + (2\zeta_7/\zeta_4)\mathbf{L}',$$

$$(\mathbf{g}')_{ijkl} = (\mathbf{g})_{ijkl} - (1/3)\delta_{ij}\delta_{kl}. \tag{62}$$

Making $\zeta_4\mathbf{C}_1^{-1}$: [eqn (59)] $-\eta_4\mathbf{C}_2^{-1}$: [eqn (60)], to get $\mathbf{e}^p$ to be expressed by other quantities, and substituting it into one of eqns (59) or (60), we obtain the following constitutive equation:

$$\dot{\mathbf{e}}^p = \zeta_4\eta_4^2\mathbf{F}_2::(\dot{\mathbf{e}}^p\otimes\dot{\mathbf{e}}^p) - \eta_4\zeta_4^2\mathbf{F}_3::(\dot{\mathbf{e}}^p\otimes\mathring{\mathbf{T}})$$
$$+ \zeta_4^3\mathbf{F}_1::(\mathring{\mathbf{T}}\otimes\mathring{\mathbf{T}}) + \mathbf{G}_1:\mathring{\mathbf{T}} + \mathbf{G}_2:\mathbf{e}^p + \hat{\mathbf{T}}, \tag{63}$$

$$\begin{aligned}
\hat{\mathbf{T}} &= \mathbf{T}_2^* - \zeta_3\mathbf{B}:\mathbf{C}_3^{-1}:\mathbf{T}^{**} + \zeta_4\mathbf{C}_2:(\mathbf{C}_3^{-1}:\mathbf{T}^{**})^2,\\
\mathbf{G}_1 &= \zeta_3\zeta_4\mathbf{G}_1^{**} - \zeta_4^2\mathbf{G}^*, \qquad \mathbf{G}_2 = -\zeta_3\eta_4\mathbf{G}_2^{**} + \zeta_4\eta_4\mathbf{G}_2^*,\\
(\mathbf{G}_1^*)_{ijkl} &= C_{2ijpq}(C_{3rquv}^{-1}C_{4prkl} + C_{3pruv}^{-1}C_{4rqkl})T_{uv}^{**},\\
\mathbf{G}_2^* &= \mathbf{G}_1^* \quad \text{with replacement } \mathbf{C}_4 \text{ by } \mathbf{C}_5; \qquad \mathbf{G}_1^{**} = \mathbf{B}:\mathbf{C}_4,\\
\mathbf{G}_2^{**} &= \mathbf{B}:\mathbf{C}_5,\\
\mathbf{C}_4 &= \mathbf{C}_3^{-1}:\mathbf{C}_1^{-1}, \qquad \mathbf{C}_5 = \mathbf{C}_3^{-1}:\mathbf{C}_2^{-1};\\
\mathbf{C}_3 &= \eta_3\zeta_4\mathbf{C}_1^{-1}:\mathbf{A} - \eta_4\zeta_3\mathbf{C}_2^{-1}:\mathbf{B},\\
\mathbf{T}_1^* &= \eta_1\mathbf{T} + \eta_2\mathbf{T}^*, \qquad \mathbf{T}_2^* = \zeta_1\mathbf{T} + \zeta_2\mathbf{T}^*;\\
\mathbf{T}^{**} &= \zeta_4\mathbf{C}_1^{-1}:\mathbf{T}_1^* - \eta_4\mathbf{C}_2^{-1}:\mathbf{T}_2^*,\\
(\mathbf{F}_1)_{ijklmn} &= C_{2ijpq}C_{4prkl}C_{4rqmn}, \qquad (\mathbf{F}_2)_{ijklmn} = C_{2ijpq}C_{5prkl}C_{5rqmn},\\
(\mathbf{F}_3)_{ijklmn} &= C_{2ijpq}(C_{5prkl}C_{4rqmn} + C_{5rqkl}C_{4prmn}),\\
(\mathbf{a}\otimes\mathbf{b})_{ijkl} &= (\mathbf{a})_{ij}(\mathbf{b})_{kl}; \qquad \{\mathbf{F}::(\mathbf{a}\otimes\mathbf{b})\}_{ij} = (\mathbf{F})_{ijklmn}(\mathbf{a})_{kl}(\mathbf{b})_{mn},
\end{aligned} \tag{64}$$

where $(\mathbf{x})_{ij\ldots}$ denotes the components of $\mathbf{x}$ in rectangular Cartesian coordinates. A certain additional condition would be required in order that eqn (63) give a unique one-to-one correspondence between $\dot{\mathbf{e}}^p$ and $\mathring{\mathbf{T}}$.

Instead of eqn (63), we can deduce other expressions for the quadratic constitutive equation by virtue of Wang's theorem[6] by considering $\dot{\mathbf{e}}^p$ as an isotropic tensor function of $\mathring{\mathbf{T}}$ and $\mathbf{T}$, or $\mathring{\mathbf{T}}$ as an isotropic tensor function of $\dot{\mathbf{e}}^p$ and $\mathbf{T}$. That is to say,

$$\dot{\mathbf{e}}^p = \gamma_1\mathbf{T} + \gamma_2[\mathbf{T}^2 - (1/3)(\mathrm{tr}\,\mathbf{T}^2)\mathbf{I}] + (\gamma_3\mathbf{g} + \gamma_4\mathbf{L}' + \gamma_5\mathbf{L}_1):\mathring{\mathbf{T}}$$
$$+ (\gamma_6\mathbf{g}' + \gamma_7\mathbf{L}'):\mathring{\mathbf{T}}^2, \tag{65}$$

$$\mathring{\mathbf{T}} = \nu_1\mathbf{T} + \nu_2[\mathbf{T}^2 - (1/3)(\mathrm{tr}\,\mathbf{T}^2)\mathbf{I}] + (\nu_3\mathbf{g} + \nu_4\mathbf{L}' + \nu_5\mathbf{L}_1):\dot{\mathbf{e}}^p$$
$$+ (\nu_6\mathbf{g}' + \nu_7\mathbf{L}'):\dot{\mathbf{e}}^{p2}, \tag{66}$$

where γ_i and ν_i ($i \neq 1, 2, \ldots, 7$) are functions of tr $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ and the fundamental invariants of $\mathbf{T}$ and $\mathring{\mathbf{T}}$, or $\mathbf{T}$ and $\dot{\mathbf{e}}^p$, respectively. However, they can be thought to be the functions of tr $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$, and the fundamental invariants of $\mathbf{T}$ and $\mathbf{e}^p$ by virtue of eqn (12).

We should note that essentially the three constitutive equations (63), (65) and (66) express the same equation. Particularly, eqns (65) and (66) express the inverse form of each other.

4. INTRODUCTION OF GENERAL INITIAL AND SUBSEQUENT ANISOTROPY

Now let us generalize our theory to the case where general initial and subsequent anisotropy plays an important role. For simplicity, we concern ourselves with the constitutive equation formulated by eqn (41). It is natural in general to think that, at the instant of initial yielding, the yield surface is smooth at any point on it, because there cannot exist any yield surface which possesses a vertex everywhere and the loading direction can be arbitrary. (For example, even for Tresca material, the yield surface is smooth except at the several specific points.) That is, the vertex will form at the onset of yielding and then develop with deformation. Therefore, at that instant, we may set $H_0 = \infty$, then eqn (41) reduces to

$$d\mathbf{e}^p = (1/2h_0)(3/2)(\mathbf{T}/\bar{\sigma}) \operatorname{tr}[(\mathbf{T}/\bar{\sigma}) \, \mathring{d}\mathbf{T}], \tag{67}$$

which is identical to

$$d\mathbf{e}^p = (1/2h_0)[(\partial\phi/\partial\boldsymbol{\sigma})/\| \partial\phi/\partial\boldsymbol{\sigma} \|] \operatorname{tr}\{[(\partial\phi/\partial\boldsymbol{\sigma})/\| \partial\phi/\partial\boldsymbol{\sigma} \|] \, \mathring{d}\mathbf{T}\}, \tag{68}$$

with the definition of ϕ as

$$\phi = \sqrt{3/2} \, (\operatorname{tr} \mathbf{T}^2)^{1/2} = \bar{\sigma}, \tag{69}$$

which can be thought as the yield function for this case, where the norm of a tensor $\mathbf{X}$ is defined as

$$\| \mathbf{X} \| = (\operatorname{tr} \mathbf{X}^2)^{1/2}. \tag{70}$$

The expression of the yield function with general initial and subsequent anisotropy, but without a vertex, is denoted by $\phi^*(\bar{\mathbf{T}})$ which expresses any yield function proposed so far, theoretically or experimentally, where

$$\bar{\mathbf{T}} = \mathbf{T} - \mathbf{R}_d, \qquad \bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \mathbf{R}, \tag{71}$$

and $\mathbf{R}_d$ is the deviatoric part of the so-called back stress $\mathbf{R}$. Referring to eqn (68), we obtain the constitutive equation without vertex effect and with the yield functions $\phi^*(\bar{\mathbf{T}})$ as follows:

$$d\mathbf{e}^p = (1/2h_0)[(\partial\phi^*/\partial\bar{\boldsymbol{\sigma}})/\| \partial\phi^*/\partial\bar{\boldsymbol{\sigma}} \|] \operatorname{tr}\{[(\partial\phi^*/\partial\bar{\boldsymbol{\sigma}})/\| \partial\phi^*/\partial\bar{\boldsymbol{\sigma}} \|] \, \mathring{d}\mathbf{T}\}. \tag{72}$$

Now let us define a generalized deviatoric stress tensor $\mathbf{T}^*$ and the strength of it σ^* as follows:

$$\mathbf{T}^*/\sigma^* = \sqrt{2/3}(\partial\phi^*/\partial\bar{\boldsymbol{\sigma}})/\| \partial\phi^*/\partial\bar{\boldsymbol{\sigma}} \|, \tag{73}$$

$$\sigma^* = \phi^*. \tag{74}$$

Then we get the equations

$$\mathbf{T}^* = \sqrt{2/3}\phi^*(\partial\phi^*/\partial\bar{\boldsymbol{\sigma}})/\| \partial\phi^*/\partial\bar{\boldsymbol{\sigma}} \|, \tag{75}$$

$$\sigma^* = \sqrt{3/2}(\operatorname{tr} \mathbf{T}^{*2})^{1/2}, \tag{76}$$

both of which have dimension of stress. In fact, $\mathbf{T}^* = \bar{\mathbf{T}}$ for $\phi^* = \phi(\bar{\mathbf{T}})$, and $\sigma^* = \bar{\sigma}$

for $\mathbf{R} = 0$. Substituting eqns (75) and (76) into eqn (72), we obtain the expression

$$d\mathbf{e}^p = (1/2h_0)(3/2)(\mathbf{T}^*/\sigma^*)\,\mathrm{tr}[(\mathbf{T}^*/\sigma^*)\,\mathring{d}\mathbf{T}], \tag{77}$$

which is formally identical to eqn (67). Extending eqn (77) to that for the case where vertex effect is considered, we finally obtain the following constitutive equation:

$$d\mathbf{e}^p = \langle P(\Theta^*)\rangle\{(1/2\sigma^*)[(1/h_0) - (1/H_0)]\mathbf{T}^*\overline{d\sigma} + (1/2H_0)\,\mathring{d}\mathbf{T}\}, \tag{78}$$

$$P(\Theta^*) = a + b\cos\Theta^*, \tag{79}$$

$$\begin{aligned}\cos\Theta^* &= \mathrm{tr}(\mathbf{T}^*\,\mathring{d}\mathbf{T})/[(\mathrm{tr}\,\mathbf{T}^{*2})(\mathrm{tr}\,\mathring{d}\mathbf{T}^2)]^{1/2}\\ &= (3/2)\,\mathrm{tr}(\mathbf{T}^*\,\mathring{d}\mathbf{T})/(\sigma^*\,\overline{d\sigma}) = (3/2)\,\mathrm{tr}[(\mathbf{T}^*/\sigma^*)\mathring{\dot{\mathbf{T}}}].\end{aligned} \tag{80}$$

Therefore, in $\boldsymbol{\Sigma}$-space defined by eqn (30), the axis of the loading surface cone is $\mathbf{T}^*$ instead of $\mathbf{T}$. The condition of continuation of plastic loading for subsequent increment $\mathring{d}\mathbf{T}$ is given by the following expressions:

$$0 \leqq \Theta^* < \Theta_{\max}; \qquad \Theta_{\max} = \cos^{-1}(-a/b). \tag{81}$$

However, another formulation is also possible. Namely, if the replacement $\mathbf{T}$ by $\mathbf{T}^*$ is considered to extend to the term $\mathring{d}\mathbf{T}$, then instead of eqn (78), we have the following constitutive equation:

$$d\mathbf{e}^p = \langle P(\hat{\Theta}^*)\rangle\{(1/2\sigma^*)[(1/h_0) - (1/H_0)]\mathbf{T}^*\,\overline{d\sigma}^* + (1/2H_0)\,\mathring{d}\mathbf{T}^*\}. \tag{82}$$

where

$$\overline{d\sigma}^* = \sqrt{3/2}(\mathrm{tr}\,\mathring{d}\mathbf{T}^{*2})^{1/2}, \tag{83}$$

which replaces $\overline{d\sigma}$, and

$$\begin{aligned}P(\hat{\Theta}^*) &= a + b\cos\hat{\Theta}^*,\\ \cos\hat{\Theta}^* &= \mathrm{tr}(\mathbf{T}^*\,\mathring{d}\mathbf{T}^*)/[(\mathrm{tr}\,\mathbf{T}^{*2})(\mathrm{tr}\,\mathring{d}\mathbf{T}^{*2})]^{1/2} = (3/2)[\mathrm{tr}(\mathbf{T}^*/\sigma^*)\mathring{\dot{\mathbf{T}}}^*].\end{aligned} \tag{84}$$

In this case, the hardening inequality is expressed as follows [cf. eqn (27)]:

$$\mathrm{tr}(d\mathbf{e}^p\,\mathring{d}\mathbf{T}^*) \geqq 0 \qquad \text{with } \mathrm{tr}(d\mathbf{e}^p\mathbf{T}^*) \geqq 0. \tag{85}$$

It is noted that the second expression in eqn (85) corresponds to eqn (20). That is, the second term of the left-hand side of eqn (20) is included in it by the replacement $\mathbf{T}$ by $\mathbf{T}^*$. Furthermore, we understand that $\boldsymbol{\Sigma}$-space is replaced by $\boldsymbol{\Sigma}^*$-space in which $\mathbf{T}^*$ plays the role of $\mathbf{T}$. In $\boldsymbol{\Sigma}^*$-space, the axis of the loading surface cone is $\mathbf{T}^*$. The condition of continuation of plastic loading is formally described by eqn (81), where the angle Θ^* is replaced by the angle $\hat{\Theta}^*$.

Finally, it is noted that the replacement of $\mathbf{T}$ and $\bar{\sigma}$ by $\mathbf{T}^*$ and σ^*, respectively, in the equations other than eqn (41) such as eqns (48), (57), (63), etc., will also lead their generalization to the case where initial and subsequent anisotropy cannot be neglected.

5. TIME-DEPENDENT INELASTIC CONSTITUTIVE EQUATIONS

5.1 *Natural time-dependent case*

If we use the natural time measure dt instead of $\overline{d\sigma}$ and $\overline{d\epsilon^p}$, we can concern ourselves with natural time-dependent inelastic constitutive equations. Then the stress- and strain-rate are defined as

$$\dot{\boldsymbol{\sigma}} = d\boldsymbol{\sigma}/dt, \qquad \dot{\boldsymbol{\epsilon}} = d\boldsymbol{\epsilon}/dt, \tag{86}$$

where d/dt denotes the so-called material time derivatives. ($\mathring{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\omega}}\boldsymbol{\sigma} + \boldsymbol{\sigma}\dot{\boldsymbol{\omega}}$.) Then eqns (13)–(18) are formally valid, where $\mathbf{e}^p$ is replaced by $\mathbf{e}^i$, i.e. total inelastic strain. However, in contrast with eqns (8) and (11), the magnitude of the norm of $\mathring{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\epsilon}}$ have no limitation. This fact characterizes the difference between plasticity and natural time-dependent inelasticity such as creep and viscoelasticity.

Eliminating $\mathbf{e}^i$ from eqn (14) and (17) as described in the earlier sections, we obtain various kinds of constitutive equation which shows dependency on the natural time.

Example 1. The simplest constitutive equation is given as follows:

$$\dot{\mathbf{e}} = \alpha_1 \mathbf{T}, \tag{87}$$

$$\alpha_1 = (3/2)(\dot{\bar{\epsilon}}/\bar{\sigma}); \quad \dot{\bar{\epsilon}} = \sqrt{2/3}(\mathrm{tr}\,\dot{\mathbf{e}}^2)^{1/2}, \quad \bar{\sigma} = \sqrt{3/2}(\mathrm{tr}\,\mathbf{T}^2)^{1/2}. \tag{88}$$

If dependency on temperature is also considered, then adopting Feltham's equation for one-dimensional case, i.e.

$$\dot{\epsilon} = \alpha \exp(-Q/kT)\sinh(\beta/kT); \quad T = \text{absolute temperature}, \tag{89}$$

and adopting the strain-hardening theory for creep, we obtain the following constitutive equation:

$$\dot{\mathbf{e}} = \left[(3/2)m^*\bar{\sigma}^{(n-m)/m}\bar{\epsilon}^{(m-1)/m} \cdot \exp\left(-\frac{Q_0}{kT}\right) \cdot \sinh\left(\frac{\beta}{kT}\right)\right]\mathbf{T}, \tag{90}$$

where m^*, m, n and β are the material constants, Q_0 is the stress-independent active energy and k is the Boltzmann's constant.

Example 2. The simplest form involving the term $\mathring{\mathbf{T}}$ is given as follows:

$$\dot{\mathbf{e}} = \alpha_1\mathbf{T} + \alpha_2\mathring{\mathbf{T}}; \quad \alpha_1 = \eta_1(\nu_1 - \alpha_2), \quad \nu_1 = \zeta_1/\eta_1. \tag{91}$$

After some manipulation, we obtain the following constitutive equation:

$$\dot{\mathbf{e}} = (3/2)\exp\left(-\frac{Q_0}{kT}\right) \cdot \sinh\left(\frac{\beta}{kT}\right) \cdot [m_1^*\bar{\sigma}^{(n-m/m}\bar{\epsilon}^{(m-1)/m} \cdot \mathbf{T} + m_2^*(\bar{\epsilon}/\bar{\sigma})\mathring{\mathbf{T}}], \tag{92}$$

$$\begin{aligned} \dot{\bar{\epsilon}} &= (2/3)[\alpha_1^2\bar{\sigma}^2 + 2\alpha_1\alpha_2\bar{\sigma}\mathring{\bar{\sigma}}\cos\Theta + \alpha_2^2\mathring{\bar{\sigma}}^2]^{1/2}, \\ \mathring{\bar{\sigma}} &= \sqrt{3/2}(\mathrm{tr}\,\mathring{\mathbf{T}}^2)^{1/2}, \\ \cos\Theta &= (3/2)\,\mathrm{tr}(\mathbf{T}\mathring{\mathbf{T}})/(\bar{\sigma}\mathring{\bar{\sigma}}), \end{aligned} \tag{93}$$

where α_1 and α_2 in eqn (93) should be found by the comparison of eqns (91) and (92). m_1^* and m_2^* are the additional material constants.

Example 3. If we add the second-rank term of $\mathbf{T}$ to eqn (91), namely

$$\dot{\mathbf{e}} = \eta_1\mathbf{T} + \eta_2\{\mathbf{T}^2 - (1/3)(\mathrm{tr}\,\mathbf{T}^2)\mathbf{I}\} + \alpha_2\mathring{\mathbf{T}} = \alpha_1\mathbf{T}^* + \alpha_2\mathring{\mathbf{T}} \text{ (say)}, \tag{94}$$

then we can formulate the following constitutive equation:

$$\dot{\mathbf{e}} = (3/2)\exp\left(-\frac{Q_0}{kT}\right)\sinh\left(\frac{\beta}{kT}\right)[m_1^*\bar{\sigma}^{(n-m)/m}\bar{\epsilon}^{(m-1)/m} \cdot\{\mathbf{T} + 3(S/\bar{\sigma})[\mathbf{T}^2 - (1/3)(\mathrm{tr}\,\mathbf{T}^2)\mathbf{I}]\} + m_2^*(\bar{\epsilon}/\bar{\sigma})\mathring{\mathbf{T}}]; \tag{95}$$

$$
\begin{aligned}
\bar{\epsilon} &= \int \dot{\epsilon}\, dt, \\
\dot{\epsilon} &= (2/3)[(\alpha_1\sigma^*)^2 + 2(\alpha_1\sigma^*)\alpha_2\mathring{\sigma} \cos\Theta^* + (\alpha_2\mathring{\sigma})^2]^{1/2} \\
\sigma^* &= \sqrt{3/2}(\mathrm{tr}\, \mathbf{T}^{*2})^{1/2}, \qquad (96) \\
\alpha_1, \alpha_2 &= [\text{to be found by comparison of eqn (94) and (95).}] \\
\cos\Theta^* &= (3/2\sigma^*\bar{\sigma})\, \mathrm{tr}(\mathbf{T}^*\mathring{\mathbf{T}}).
\end{aligned}
$$

$$
\begin{aligned}
S &= \sqrt{(\sigma_{III}/\bar{\sigma})^6 + R^{2n/m} - 1} - (\sigma_{III}/\bar{\sigma})^3 \\
\sigma_{III}^3 &= (9/2)\,\mathrm{tr}\,\mathbf{T}^3 = (27/2)\mathrm{III}_\mathbf{T}, \quad \mathrm{III}_\mathbf{T} = \det(\mathbf{T}) \\
R^2 &= 1 + 4(R_0^2 - 1)\cos^2\{(2\pi/3) - (1/3)\cos^{-1}(\sigma_{III}/\bar{\sigma})^3\} \qquad (97) \\
R_0 &= \begin{cases} R_t = a(\bar{\epsilon})/b(\bar{\epsilon}) & \text{for } 0 \leqq (\sigma_{III}/\bar{\sigma}) \leqq 1, \\ R_c = a(\bar{\epsilon})/c(\bar{\epsilon}) & \text{for } -1 \leqq (\sigma_{III}/\bar{\sigma}) < 0, \end{cases} \\
a(\bar{\epsilon}), b(\bar{\epsilon}), c(\bar{\epsilon}) &=
\end{aligned}
$$

[($\bar{\sigma}$ vs $\bar{\epsilon}$) relations for pure shear, uniaxial tension and uniaxial compression, respectively], where, following Ohashi *et al.*[9], the effect of the third invariant of stress tensor σ_{III} is taken into consideration.

5.2 *Natural and internal time-dependent case*

The most general time measure is composed of dt and $\overline{d\sigma}$ or $\overline{d\epsilon}$, where $\overline{d\epsilon}$ denotes the inelastic strain increment measure. The manner of the combination is not definite. Valanis considered a combination of dt and $\overline{d\epsilon}$ in his so-called endochronic theory of inelasticity[10]. Here we adopt the following general time measure:

$$\overline{\overline{d\sigma}}^2 = \overline{d\sigma}^2 + a_1\, dt^2, \qquad a_1 \geqq 0, \tag{98}$$

or

$$\overline{\overline{d\epsilon}}^2 = \overline{d\epsilon}^2 + a_2\, dt^2, \qquad a_2 \geqq 0. \tag{99}$$

We can express the constitutive equation of viscoplasticity by making use of this time measure. Deviatoric stress- and strain-rates are defined as

$$\mathring{\mathbf{T}} = \mathring{d}\mathbf{T}/\overline{\overline{d\sigma}}, \qquad \dot{\mathbf{e}} = d\mathbf{e}/\overline{\overline{d\epsilon}}, \tag{100}$$

and the following additional definitions will also be used:

$$\mathring{\sigma} = \overline{d\sigma}/\overline{\overline{d\sigma}}, \qquad \dot{\epsilon} = \overline{d\epsilon}/\overline{\overline{d\epsilon}}. \tag{101}$$

Again eqns (13)–(18) are formally valid. Taking the identity $\mathrm{tr}(\mathring{d}\mathbf{T}/\overline{d\sigma})^2 = 2/3$ and $\mathring{\mathbf{T}} = (\mathring{d}\mathbf{T}/\overline{d\sigma})\mathring{\sigma}$ into consideration, from eqn (14), we obtain the following scalar relation between $\mathbf{T}$ and $\mathbf{e}$ which corresponds to eqn (19):

$$\mathring{\sigma} = \hat{\phi}_1[\mathrm{tr}\,\boldsymbol{\sigma}, \mathrm{tr}\,\mathbf{T}^2, \ldots, \mathrm{tr}(\mathbf{T}^2\mathbf{e}^2)], \tag{102}$$

where

$$\mathring{\sigma} = \overset{*}{\sigma}/\sqrt{\overset{*}{\sigma}{}^2 + a_1}, \tag{103}$$

$$\overset{*}{\sigma} = \overline{d\sigma}/dt = \sqrt{3/2}[\mathrm{tr}(\mathring{d}\mathbf{T}/dt)^2]^{1/2} \qquad \text{(definition)}. \tag{104}$$

That is, $\overset{*}{\sigma}$ denotes the strength of stress-rate tensor based on the natural time t. Therefore, eqn (102) expresses the yield condition which involves the dependency on the stress-rate based on the natural time for the case where the stress-rate is controlled.

On the other hand, the identity $\mathrm{tr}(\mathrm{d}\mathbf{e}/\overline{\mathrm{d}\epsilon})^2 = 3/2$ and $\dot{\mathbf{e}} = (\mathrm{d}\mathbf{e}/\overline{\mathrm{d}\epsilon})\dot{\epsilon}$ yield the following scalar relation between T and e from eqn (17):

$$\dot{\epsilon} = \hat{\phi}_2[\mathrm{tr}\,\boldsymbol{\sigma}, \mathrm{tr}\,\mathbf{T}^2, \ldots, \mathrm{tr}(\mathbf{T}^2\mathbf{e}^2)], \tag{105}$$

where

$$\dot{\epsilon} = \overset{*}{\epsilon}/\sqrt{\overset{*}{\epsilon}{}^2 + a_2}; \qquad \overset{*}{\epsilon} = \overline{\mathrm{d}\epsilon}/\mathrm{d}t = \sqrt{2/3}[\mathrm{tr}(\mathrm{d}\mathbf{e}/\mathrm{d}t)^2]^{1/2} \qquad \text{(definition)}. \tag{106}$$

Equation (105) expresses the yield condition which involves the dependency on the strain-rate, based on the natural time for the case where the strain-rate is controlled. The factors a_1 and a_2, and the functions $\hat{\phi}_1$ and $\hat{\phi}_2$ can be temperature dependent.

Equation (102) and (105) are rewritten as follows:

$$A_\sigma\mathring{\sigma} + \kappa_\sigma = \phi_1[\mathrm{tr}\,\boldsymbol{\sigma}, \mathrm{tr}\,\mathbf{T}^2, \ldots, \mathrm{tr}(\mathbf{T}^2\mathbf{e}^2)], \tag{107}$$

$$A_\epsilon\dot{\epsilon} + \kappa_\epsilon = \phi_2[\mathrm{tr}\,\boldsymbol{\sigma}, \mathrm{tr}\,\mathbf{T}^2, \ldots, \mathrm{tr}(\mathbf{T}^2\mathbf{e}^2)], \tag{108}$$

where A_σ, A_ϵ, κ_σ and κ_ϵ may be the functions of ($\mathrm{tr}\,\boldsymbol{\sigma}$, $\mathrm{tr}\,\mathbf{e}^2$, $\mathrm{tr}\,\mathbf{e}^3$), (and/or temperature T), or constants. Evidently, the equations

$$A_\sigma + \kappa_\sigma = \phi_1 \qquad \text{and} \qquad A_\epsilon + \kappa_\epsilon = \phi_2 \tag{109}$$

are the ultimate dynamic yield conditions for the case where $\overset{*}{\sigma}$ or $\overset{*}{\epsilon}$ tends to infinity, respectively. On the other hand, the equations

$$\kappa_\sigma = \phi_1 \qquad \text{and} \qquad \kappa_\epsilon = \phi_2 \tag{110}$$

are the completely statical yield conditions for the case where $\overset{*}{\sigma}$ or $\overset{*}{\epsilon}$ tends to 0, respectively. $\kappa_\sigma = 0$, or $\kappa_\epsilon = 0$, implies that the threshold value of the static yield stress is equal to 0.

Example 1. The simplest constitutive equation for this case is given as follows:

$$\dot{\mathbf{e}} = \alpha_1\mathbf{T}, \qquad \alpha_1 = (3/2)(\dot{\epsilon}/\bar{\sigma}). \tag{111}$$

Noting eqns (106) and (108) with assumption $A_\epsilon > 0$, we get

$$\dot{\mathbf{e}} = (3/2)(\langle\phi_2 - \kappa_\epsilon\rangle/A_\epsilon\bar{\sigma})\mathbf{T} = (1/2\eta')\left\langle\frac{\phi_2}{\sqrt{3J_2}} - \frac{\kappa_s}{\sqrt{J_2}}\right\rangle\mathbf{T}, \tag{112}$$

where

$$1/\eta' = 3/A_\epsilon, \qquad \bar{\sigma} = \sqrt{3J_2}, \qquad \kappa_s = \kappa_\epsilon/\sqrt{3},$$

and J_2 = the second invariant of $\boldsymbol{\sigma}$. For Mises material, i.e. $\phi_2 = \sqrt{3J_2} = \bar{\sigma}$,

$$\dot{\mathbf{e}} = (1/2\eta')\left\langle 1 - \frac{\kappa_s}{\sqrt{J_2}}\right\rangle\mathbf{T}, \tag{113}$$

and κ_s is the static yield stress. Because $\dot{\mathbf{e}} = (\mathrm{d}\mathbf{e}/\mathrm{d}t)/\sqrt{\overset{*}{\epsilon}{}^2 + a_2}$, eqn (113) is rewritten as follows:

$$\mathrm{d}\mathbf{e}/\mathrm{d}t = (1/2\eta)\left\langle 1 - \frac{\kappa_s}{\sqrt{J_2}}\right\rangle\mathbf{T}, \qquad \eta = \eta'/\sqrt{\overset{*}{\epsilon}{}^2 + a_2}, \tag{114}$$

which is exactly equivalent to the so-called Sokolovski–Malvern constitutive equation for dynamic plasticity[11].

Example 2. The simplest constitutive equation involving the term $\mathring{\mathbf{T}}$ is obtained as follows:

$$\dot{\mathbf{e}} = \alpha_1 \mathbf{T} + \alpha_2 \mathring{\mathbf{T}}. \tag{115}$$

Some manipulation leads to the following expression:

$$\begin{aligned} \mathrm{d}\mathbf{e}/\mathrm{d}t &= \left(\frac{1}{2G} + \frac{\langle P_\mathrm{d}(\Theta)\rangle}{2H_s}\right)(\mathrm{d}\mathring{\mathbf{T}}/\mathrm{d}t) + \frac{\langle P_\mathrm{d}(\Theta)\rangle}{2\bar{\sigma}\eta_\mathrm{d}}(\phi_1 - \kappa_\sigma)\mathbf{T}, \\ \mathrm{tr}(\mathrm{d}\boldsymbol{\epsilon}/\mathrm{d}t) &= (1/3K)\,\mathrm{tr}(\mathrm{d}\boldsymbol{\sigma}/\mathrm{d}t) + 3\beta\dot{T}\mathbf{I}, \end{aligned} \tag{116}$$

where isotropic elasticity is taken into consideration, i.e. G = rigidity modulus, K = bulk modulus, and

$$\begin{aligned} \dot{T} &= \mathrm{d}T/\mathrm{d}t, \\ P_\mathrm{d}(\Theta) &= \left(1 - p\cdot\frac{1-\cos\Theta}{1-\cos\Theta_0}\right)\times\gamma, \\ \gamma &= 1 \text{ for plastic state}, \\ &= 0 \text{ for elastic state}, \\ \Theta_0 &= \cos^{-1}[-h_s/(H_s - h_s)], \\ 3h_s &= \overline{\mathrm{d}\sigma}/\overline{\mathrm{d}\epsilon} \quad \text{for static proportional loading}, \\ H_s &= \text{instantaneous static vertex-hardening rate}, \\ p &= \text{a material constant}, \quad (1/2)(1-\cos\Theta_0) \geqq p \geqq 0, \\ 1/\eta_\mathrm{d} &= (\sqrt{\overset{*}{\sigma}{}^2 + a_1}/A_\sigma)\left(\frac{1}{h_s} - \frac{1}{H_s}\right), \\ \beta &= \text{linear thermal expansion coefficient}, \end{aligned} \tag{117}$$

where dependency on temperature is also considered.

6. CONCLUSION

From the viewpoint of tensor algebra, a class of the plastic and other inelastic constitutive equations is formulated which expresses generally one-to-one correspondence between inelastic strain-rate and stress-rate together with appropriate history-dependency and general initial and subsequent anisotropy. Further discussions and concrete applications of simpler plastic constitutive equations among them will soon appear in the following works of this series.

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